7. REPRESENTATIONS OF FINITE GROUPS

§ 7.1. Review of Relevant Linear Algebra

As we saw earlier, every finite group is isomorphic to a group of matrices. The advantage of the matrix disguise is that we can use all the resources of linear algebra. So to help us study a particular group, G, we consider matrix groups that are isomorphic to (or more generally, homomorphic images of) G.

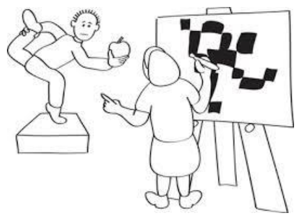
But first a quick review of linear algebra. You remember that square matrices have **determinants** and that the determinant of a product is the product of the determinants. They also have **eigenvalues** and these are related to the determinant in that the determinant of a matrix is the product of its eigenvalues. The sum of the eigenvalues is another important quantity associated with a matrix – it's the trace.

The **trace** of a matrix is simply the sum of its diagonal entries. The off-diagonal components are ignored. With so much information thrown away it's amazing that it's of much use. In linear algebra trace is hardly mentioned. We learn that trace is equal to the sum of the eigenvalues and so it's a useful little check on eigenvalue calculations. That's all. But for representation theory it will become our most valuable tool!

You'll need to recall that **similar matrices** have the same eigenvalues, and hence the same determinant and, most importantly, the same trace. Remember too that if a matrix satisfies a polynomial equation the eigenvalues satisfy the same equation. Since we'll be dealing with finite groups the matrices that will arise in our representations will satisfy equations of the form $A^m = I$. The eigenvalues will therefore be roots of unity.

Associated with eigenvalues are **eigenvectors**. A matrix is **diagonalisable** (similar to a diagonal matrix) if and only if it there's a basis of eigenvectors. Matrices of finite order are diagonalisable so all the matrices that arise in the representation of finite groups will be diagonalisable.

Finally you'll need to know a little of the theory of inner product spaces. These are vector spaces with an inner product $\langle \mathbf{u} \mid \mathbf{v} \rangle$ satisfying certain axioms. The square of the **length** of a vector **u** is $\langle \mathbf{u} \mid \mathbf{u} \rangle$ and a **unit** vector is one whose length is 1. Two vectors **u**, **v** are **orthogonal** if $\langle \mathbf{u} \mid \mathbf{v} \rangle = 0$ and an **orthonormal basis** is a basis of mutually orthogonal unit vectors.



§ 7.2. Representations

A **representation** of **degree** *n* of a group G over the field F is defined to be a homomorphism $\rho: G \to GL(n, F)$ for some *n*. By the first isomorphism theorem the image of a representation ρ is a group of $n \times n$ matrices that's isomorphic to the quotient group $G/\ker(\rho)$.

A **linear representation** is a representation of degree 1. This is an important special case. Of course a 1 × 1 matrix behaves like its one and only component so a linear representation is essentially a homomorphism to $F^{\#}$, the group, under multiplication, of the non-zero elements of the field F.

Among the linear representations is the so-called trivial representation. The **trivial representation** is τ(g) = 1 for all $g \in G$. Not very exciting perhaps, but the trivial representation is as important to representation theory as the number 0 is to arithmetic or the empty set to set theory. The trivial representation squeezes the group entirely into one element so that no information about the group remains. The kernel of the trivial representation is the whole group. At the other end of the spectrum are the faithful representations.

A representation is **faithful** if its kernel is trivial. The image under a faithful representation is isomorphic to the group itself. It might seem that these are the best representations because they don't lose any information. But a suitable collection of unfaithful representations is usually more useful.

Example 1:

The following are some of the representations of the Klein Group $\mathbf{V}_4$ with presentation

$$\langle A, B \mid A^2, B^2, AB = BA\rangle.$$

To begin with there's the trivial representation:

$$\tau(1) = 1,\ \tau(A) = 1,\ \tau(B) = 1,\ \tau(AB) = 1.$$

Then there are three other linear representations. Since every element of the group satisfies $g^2 = 1$, a linear representation must map each element to a complex number satisfying $x^2 = 1$. So the linear representations of G are:

	1	A	B	AB
τ	1	1	1	1
α	1	1	–1	–1
β	1	–1	1	–1
γ	1	–1	–1	1

Then there's a faithful representation that maps A to $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ B to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and AB to $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

If G is a group of permutations we can represent each element g by the corresponding permutation matrix, (a_{ij}) where:

$$a_{ij} = \begin{cases} 1 \text{ if } g(i) = j \\ 0 \text{ if } g(i) \neq j \end{cases}$$

Such a representation is called a **permutation representation.** It will always be faithful.

Cayley's theorem shows that every finite group can be considered as a group of permutations on itself since, for $g \in$ G, the map $x \to xg$ is a permutation $\pi(g)$ of G and π is a homomorphism. If G has order n we can represent $\pi(g)$ by an $n \times n$ permutation matrix. This permutation representation is called the **regular representation**.

Example 2: The regular representation for $\mathbf{V}_4$ above is:

$$I \to \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix},\ A \to \begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{pmatrix},\ B \to \begin{pmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{pmatrix},\ AB \to \begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{pmatrix}.$$

Two representations ρ, σ are **equivalent** if there's an invertible matrix S such that:

$$\rho(g) = S^{-1}\,\sigma(g)\,S \text{ for all } g \in G.$$

Example 3:

The representations of $\langle A \mid A^3 = 1\rangle$ include:

	I	**A**	**A^2**
ρ_1	1	1	1
ρ_2	1	ω	ω^2
ρ_3	$\begin{pmatrix} 1&0 \\ 0&1 \end{pmatrix}$	$\begin{pmatrix} 0&-1 \\ 1&-1 \end{pmatrix}$	$\begin{pmatrix} -1&1 \\ -1&0 \end{pmatrix}$
ρ_4	$\begin{pmatrix} 100 \\ 010 \\ 001 \end{pmatrix}$	$\begin{pmatrix} 010 \\ 001 \\ 100 \end{pmatrix}$	$\begin{pmatrix} 001 \\ 100 \\ 010 \end{pmatrix}$

ρ_1 is the trivial representation.

ρ_1, ρ_2 are linear representations.

$\rho_2 - \rho_4$ are faithful representations.

ρ_4 is the regular representation.

Example 4:

The representations of $\mathbf{S}_3$ include:

	I	**(123)**	**(132)**	**(12)**	**(13)**	**(23)**
ρ_1	1	1	1	1	1	1
ρ_2	1	1	1	–1	–1	–1
ρ_3	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&-1\end{pmatrix}$	$\begin{pmatrix}-1&1\\-1&0\end{pmatrix}$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$	$\begin{pmatrix}1&-1\\0&-1\end{pmatrix}$	$\begin{pmatrix}-1&0\\-1&1\end{pmatrix}$
ρ_4	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$
ρ_5	$\begin{pmatrix}10\\01\end{pmatrix}$	$\begin{pmatrix}\omega&0\\0&\omega^2\end{pmatrix}$	$\begin{pmatrix}\omega^2&0\\0&\omega\end{pmatrix}$	$\begin{pmatrix}01\\10\end{pmatrix}$	$\begin{pmatrix}0&\omega^2\\\omega&0\end{pmatrix}$	$\begin{pmatrix}0&\omega\\\omega^2&0\end{pmatrix}$
ρ_6	$\begin{pmatrix}100\\010\\001\end{pmatrix}$	$\begin{pmatrix}010\\001\\100\end{pmatrix}$	$\begin{pmatrix}001\\100\\010\end{pmatrix}$	$\begin{pmatrix}010\\100\\001\end{pmatrix}$	$\begin{pmatrix}001\\010\\100\end{pmatrix}$	$\begin{pmatrix}100\\001\\010\end{pmatrix}$

where $\omega = e^{2\pi i/3}$. Also the following, ρ_7:

I	**(123)**	**(132)**
$\begin{pmatrix}100000\\010000\\001000\\000100\\000010\\000001\end{pmatrix}$	$\begin{pmatrix}010000\\001000\\100000\\000010\\000001\\000100\end{pmatrix}$	$\begin{pmatrix}001000\\100000\\010000\\000001\\000100\\000010\end{pmatrix}$

(12)	**(13)**	**(23)**
$\begin{pmatrix}000100\\000001\\000010\\100000\\001000\\010000\end{pmatrix}$	$\begin{pmatrix}000010\\000100\\000001\\010000\\100000\\001000\end{pmatrix}$	$\begin{pmatrix}000001\\000010\\000100\\001000\\010000\\100000\end{pmatrix}$

- ρ_1, ρ_2 are linear representations; ρ_3, ρ_4 and ρ_5 have degree 2, ρ_6 has degree 3 and ρ_7 has degree 6.
- ρ_1 is the trivial representation.
- ρ_6 and ρ_7 are permutation representations.
- ρ_7 is the regular representation.
- ρ_3, ρ_5, ρ_6 and ρ_7 are faithful.
- ρ_3 is equivalent to ρ_5 since $S^{-1}\rho_3(g)S = \rho_5(g)$ where $S = \begin{pmatrix} 1 & 1 \\ \omega & 1+\omega \end{pmatrix}$.

§ 7.3. Characters of Groups

As rich as matrices are, they're a little too bulky. So instead of considering the matrices themselves we consider their traces.

The **trace** of a matrix is the sum of the diagonal components so it's a very easy quantity to calculate – much easier than determinants or eigenvalues. But it's closely related to eigenvalues in that the **trace of a matrix is the sum of the eigenvalues**. And **similar matrices have the same trace**.

The **character** over a field F of a representation ρ of a finite group G is the map χ: G $\to$ F defined by:

$$\chi(g) = \text{trace } \rho(g).$$

Example 5:
The character of ρ_5, in Example 3, is:

	I	(123)	(132)	(12)	(13)	(23)
χ_5	2	–1	–1	0	0	0

Remember that $1 + \omega + \omega^2 = 0$.

Concepts such as ‘degree’, ‘faithful’ ‘regular’ and ‘trivial’ extend to characters. So the regular character of $\mathbf{S}_3$ is the character of the regular representation, ρ_7, in example 2. It is:

	I	(123)	(132)	(12)	(13)	(23)
χ	6	0	0	0	0	0

Theorem 1:
(1) Equivalent representations have the same character.
(2) Characters are constant on conjugacy classes.
Proof: Both of these follow from the fact that similar matrices have same trace. For example if the representation ρ is equivalent to σ then there exists an invertible matrix S such that $\rho(g) = S^{-1}\sigma(g)S$ and, being similar, these have the same trace. ☺✋

We can easily read off the degree of a character (meaning the degree of the corresponding representation) by simply looking at its value on 1.

Theorem 2: The degree of a character χ is $\chi(1)$.
Proof: If ρ is the representation of degree n that corresponds to the character χ then $\rho(1)$ is the $n \times n$ identity matrix whose trace is n. ☺✋

There's a shortcut we can use for permutation representations. We can pass directly from the permutations to the character without having to think about the matrices.

Theorem 3: If χ is a permutation character, $\chi(g)$ is the number of symbols fixed by g.
Proof: If ρ is the permutation representation itself then the i-j entry of $\rho(g)$ is 1 if $g(i) = j$ and it is 0 otherwise so $\chi(g)$ is simply the number of 1's on the diagonal. ☺✋

Example 6: If G = $\mathbf{S}_4$ and χ is the permutation character, $\chi((123)) = 1$, since (123) fixes 1 element, $\chi((12)) = 2$, $\chi(I) = 4$ and $\chi((1234)) = 0$.

Theorem 4: If χ is a character of G over $\mathbb{C}$ of degree n and $g \in G$ has order m then $\chi(g)$ is a sum of n numbers, each of which is an m^{th} root of 1.
Proof: If $g^m = 1$ and ρ is the corresponding representation then $\rho(g)^m$ is the $n \times n$ identity matrix I. The matrix $\rho(g)$ is thus an $n \times n$ matrix and so has n eigenvalues over $\mathbb{C}$. Each of these must satisfy the equation $\lambda^m = 1$ and so be an m^{th} root of 1. ☺✋

Example 7: If g is an element of order 2 and χ is a character of degree 3, corresponding to the representation ρ, then the eigenvalues of ρ will be ± 1. So $\chi(g) \in \{3, 1, -1, -3\}$. If g has order 3 and χ is a character of degree 2, corresponding to the representation ρ, then the eigenvalues of ρ will be two values chosen from 1, ω, ω^2, with possible repetitions. The possibilities for $\chi(g)$ are thus 2, 2ω, $2\omega^2$, $1 + \omega = -\omega^2$, $1 + \omega^2 = -\omega$ and $\omega + \omega^2 = -1$.

Theorem 5: The characters, over $\mathbb{C}$, of an element of finite order and its inverse are complex conjugates.

Proof: The eigenvalues of $\rho(g^{-1})$ are the inverses of those for $\rho(g)$. But these eigenvalues are roots of unity and so lie on the unit circle. Hence their inverses are the same as their conjugates. And the sum of these conjugates is the conjugate of the sum. ☺🖐

Example 8: If χ is a character of a group of permutations and $\chi((1234)) = 1 + 3i$ then $\chi((1432)) = 1 - 3i$.

Theorem 6: If χ is the character of a representation ρ over $\mathbb{C}$ of degree n of a finite group G and $g \in$ G then

$$|\chi(g)| \le n.$$

Proof: If the eigenvalues of $\rho(g)$ are $\lambda_1, \lambda_2, \ldots, \lambda_n$ then $|\lambda_1 + \lambda_2 + \ldots \lambda_n| \le |\lambda_1| + |\lambda_2| + \ldots |\lambda_n| = n$. ☺🖐

Theorem 7: $\chi(g) = \deg \chi$ if and only if $g \in \ker(\rho)$.
Proof: Let $n = \deg \chi$. The sum of n roots of 1 is equal to n if and only if they're all 1. If ρ is a corresponding representation then $\rho(g)$, is diagonalisable with all its eigenvalues equal to 1 and so must be the identity matrix. ☺🖐

Example 9: The characters of the above representations ρ_1 to ρ_7 of S_3 are:

	I	**(123)**	**(132)**	**(12)**	**(13)**	**(23)**
χ_1	1	1	1	1	1	1
χ_2	1	1	1	–1	–1	–1
χ_3	2	–1	–1	0	0	0
χ_4	2	2	2	0	0	0
χ_5	2	–1	–1	0	0	0
χ_6	3	0	0	1	1	1
χ_7	6	0	0	0	0	0

Example 10: The characters of $\mathbf{S}_4$ include the following. (Since all permutations with a given cycle structure are conjugate they have the same characters, so we need only list the characters by cycle structure.)

I	**(××)**	**(×××)**	**(××××)**	**(××)(××)**	
1	1	1	1	1	trivial
1	–1	1	–1	1	odd/even
4	2	1	0	0	permutation
24	0	0	0	0	regular

§ 7.4. Class Functions

A **class function** for G over a field F is a map: G→F which is constant on conjugacy classes.

Example 11: Some class functions for S_3 are:

I	(12)	(13)	(23)	(123)	(132)
17	–5	–5	–5	π	π
–42	¾	¾	¾	1+i	1+i
1	1	1	1	1	1

Theorem 8: The set of class functions of a group G over a field F is a vector space CF(G, F) over F and its dimension over F is the number of conjugacy classes of G.

Proof: It's easily checked that the class functions form a vector space under the usual operations. A basis is the set of class functions, which take the value 1 on some conjugacy class and 0 on the others. The number of these is the number of conjugacy classes. ☺✋

Example 12: A basis for the space of class functions of S_3 over ℂ is:

	I	(12)	(13)	(23)	(123)	(132)
$\boldsymbol{e_1}$	1	0	0	0	0	0
$\boldsymbol{e_2}$	0	1	1	1	0	0
$\boldsymbol{e_3}$	0	0	0	0	1	1

The class functions given in Example 11 are expressible (uniquely) as linear combinations of e_1, e_2, e_3 as:

$$17e_1 - 5e_2 + \pi e_3;$$
$$-42e_1 + \tfrac{3}{4}e_2 + (1 + i)e_3 \text{ and}$$
$$e_1 + e_2 + e_3.$$

A character χ is **reducible** if $\chi = \Psi + \Omega$ for some characters Ψ, Ω. If not, it is **irreducible**. Irreducible characters are the basic building blocks of group characters.

Theorem 9: Linear characters are irreducible.
Proof: Suppose χ is linear. If $\chi = \Psi + \Omega$ for characters Ψ and Ω, then $\deg \chi = \deg \Psi + \deg \Omega \geq 2$, a contradiction. ☺🖐

Theorem 10: Every character is a sum of irreducible characters.
Proof: We prove this by induction on the degree of a character. If χ is reducible, $\chi = \Psi + \Omega$ for characters Ψ, Ω. By induction, each is a sum of irreducible characters and hence so too is χ. ☺🖐

Certainly if a character is linear we know that it's irreducible. But there are irreducible characters of larger degrees. For example χ_3 in Example 9 is irreducible. How can we know this? After all it can be broken up as the sum of the two class functions Ψ and Ω.

	I	(123)	(132)	(12)	(13)	(23)
χ_3	2	–1	–1	0	0	0
Ψ	1	–1 + i	–1 + i	1	1	1
Ω	1	–i	–i	–1	–1	–1

How do we know that Ψ and Ω aren't characters? That's not difficult because if Ψ was a character –1 + i would have to be a cube root of 1.

But how do we know that there isn't some other decomposition in which the pieces are both characters? The answer is to make the space of class functions into an inner product space.

From now on we will be doing what is called **ordinary representation theory**. This simply means that the field over which we operate is ℂ, the field of complex numbers. One can do representation theory over other fields but sometimes things don't go as nicely as they do over ℂ. There are three reasons. Finite fields involve primes that can give problems if they divide the group order. In ℂ no element (except the identity) has finite additive order, or to use technical terminology, ℂ has 'characteristic zero'. But ℝ and ℚ also have characteristic zero. What's wrong with them? Their trouble is that they're not algebraically closed. We may get matrices that fail to have eigenvalues in ℝ or in ℚ, which makes life more complicated. The third reason why ℂ works so beautifully is that we can exploit the concept of complex conjugates.

We make CF(G, ℂ) into an inner product space by defining the inner product of two class functions by

$$\langle\chi|\Psi\rangle = \frac{1}{|G|}\sum_{g\in G}\chi(g)\overline{\Psi(g)}.$$

Example 13: In the above table all three rows represent class functions.

$$\langle\chi_3|\Psi\rangle = \frac{1}{6}[2 + 2(-1)(-1-i) + 0]$$

$$= \frac{1}{6}(4 + 2i) \text{ and}$$

$$\langle\Omega|\Omega\rangle = \frac{1}{6}[1 + 2(-i)(i) + 3(-1)(-1)] = 1.$$

§ 7.5. The Fundamental Theorem of Characters

Theorem 11: (Fundamental Theorem of Characters) The irreducible characters of a finite group G over ℂ form an orthonormal basis for CF(G, ℂ). ☺

We won't be proving this theorem. To do so would take us far from group theory into ring theory and the theory of modules over non-commutative rings. Instead we'll examine the consequences of this important theorem.

Theorem 12: The number of irreducible characters of a finite group G, over ℂ is equal to the number of conjugacy classes of G.

Proof: We saw already that the dimension of CF(G, ℂ) over ℂ is the number of conjugacy classes. ☺✋

Theorem 13: If χ, Ψ are distinct irreducible characters of a finite group G over ℂ:

$$\sum_{g \in G} \chi(g)\overline{\Psi(g)} = 0 .$$

Proof: Distinct irreducible characters are orthogonal class functions. ☺✋

Theorem 14: If χ is an irreducible character of a finite group G over ℂ then:

$$\sum_{g \in G} |\chi(g)|^2 = |G| .$$

Proof: Irreducible characters have unit length. ☺✋

Theorem 15: Suppose G is a finite group with irreducible characters $\chi_1, \ldots, \chi_n$ over ℂ. If χ is any character, expressible as a sum of irreducible characters by

$\chi = \Sigma m_i \chi_i$ then:

(1) for each i, $m_i = \langle \chi_i \mid \chi_i \rangle$;

(2) $\langle \chi | \chi \rangle = \Sigma m_i^2$.

Proof:

(1) $\langle \chi | \chi_i \rangle = \Sigma m_j \langle \chi_j | \chi_i \rangle = m_i \langle \chi_i | \chi_i \rangle = m_i$.

$$(2)\quad \langle\chi|\chi\rangle = \sum_{i,j=1}^{n} m_i m_j \langle\chi_i \mid \chi_j\rangle$$

$$= \sum_{i=1}^{n} m_i^2 \langle\chi_i \mid \chi_i\rangle \text{ by orthogonality}$$

$$= \sum_{i=1}^{n} m_i^2 \text{ since each } |\chi_i| = 1.$$ ☺✋

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Corollary: A character χ is irreducible if and only if

$$\langle\chi \mid \chi\rangle = 1.$$

Theorem 16: If Φ is the regular character and $\chi_1, \ldots, \chi_k$ are the irreducible characters with degrees $n_1, \ldots, n_k$ then

$$\Phi = \Sigma n_i\chi_i.$$

Proof: $\Phi(g) = |G|$ if $g = 1$ and 0 otherwise.
So $\langle\Phi \mid \chi_i\rangle = n_i$. ☺✋

Theorem 17: If $\chi_1, \ldots, \chi_k$ are the irreducible characters with degrees $n_1, \ldots, n_k$ then $\Sigma n_i^2 = |G|$.
Proof: If Φ is the regular character, $\langle\Phi \mid \Phi\rangle = |G| = \Sigma n_i^2$.
☺✋

§ 7.6. Character Tables

The **character table** for a finite group G, over ℂ, gives the value of each irreducible character on each conjugacy class. Denote the value of χ_i on an element of the conjugacy class Γ_j by $\chi_i(\Gamma_j)$ or more simply as χ_{ij}. Because the number of irreducible characters is equal to the number of conjugacy classes the table is square. It's also useful to record the sizes of the conjugacy classes and the orders of their elements as extra parts of the character table.

class	$\Gamma_1 = \{1\}$	Γ_2	...	Γ_k
size	**1**	h_2	...	h_k
χ_1	1	1	...	1
χ_2	n_2	χ_{22}	...	χ_{2k}
...	...	...	...	...
χ_k	n_k	χ_{k2}	...	χ_{kk}
order	**1**	m_2	...	m_k

Theorem 18: The character table of a finite group G, over ℂ, has the following properties:

(1) $\sum_{i=1}^{k} h_i = |G|$;

(2) $\sum_{i=1}^{k} n_i^2 = |G|$;

(3) $\sum_{t=1}^{k} h_t \chi_{it} \overline{\chi_{jt}} = \begin{cases} 0 \text{ if } i \neq j \\ |G| \text{ if } i = j \end{cases}$ (orthogonality of the rows);

(4) $\sum_{t=1}^{k} \chi_{ti} \overline{\chi_{tj}} = \begin{cases} 0 \text{ if } i \neq j \\ \frac{|G|}{h_i} \text{ if } i = j \end{cases}$ (orthogonality of the columns).

(Here χ_{ij} is the value of the irreducible character χ_i on the elements of the conjugacy class Γ_j,

n_i is the degree of χ_i and

h_j is the size of Γ_j.)

Proof: (1) is just the class equation;

(2) is theorem 17;

(3) is the fundamental theorem of characters (Theorem 11).

If A is the matrix (a_{ij}) where $a_{ij} = \sqrt{h_j/|G|}\, \chi_{ij}$ then (3) implies that A is a unitary matrix, that is A A* = I where A* is the conjugate transpose of A.
It follows that A*A = I which gives (4). ☺✋

Example 14: The following is the character table for a certain group G.

class	$\mathbf{\Gamma_1}$	$\mathbf{\Gamma_2}$	$\mathbf{\Gamma_3}$	$\mathbf{\Gamma_4}$
size	**1**	**3**	**4**	**4**
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	–1	0	0
order	**1**	**2**	**3**	**3**

Here ω and ω^2 are the two non-real cube roots of unity. Remember that they're conjugates of one another. And never forget that $1 + \omega + \omega^2 = 0$. That often comes in handy!

We can see that $|G| = 1 + 3 + 4 + 4 = 12$. Note too that $12 = 1^2 + 1^2 + 1^2 + 3^2$, the sum of squares of the degrees of the irreducible characters.

G has 3 elements of order 2 (in Γ_2) and 8 of order 3 (in each of Γ_3 and Γ_4).

Note the orthogonality of the rows. For example with χ_2 and χ_3 we get:

$1.1 + 3.1.1 + 4.\omega.\omega + 4.\omega^2.\omega^2 = 4 + 4\omega^2 + 4\omega = 0$. And the sums of squares of the moduli of the entries along each row (suitably weighted by the class sizes) are all 12. Along the second row it is:

$$1^2 + 3(1^2) + 4|\omega|^2 + 4|\omega^2| = 12$$

and along the third row it is:

$$3^2 + 3.1^2 + 4.0 + 4.0 = 12.$$

Note the orthogonality of the columns. For example taking the 3rd and 4th columns we get $1.1 + \omega.\omega + .\omega^2.\omega^2 = 1 + \omega^2 + \omega = 0$. Taking the sum of squares of the moduli down each column you get the order of the group, 12, divided by the class size. For example, down column 2 we get $1 + 1 + 1 + 1 = 4 = 12/3$ and down column 4 we get $1 + 1 + 1 + 0 = 3 = 12/4$.

The "–1" entry is the trace of a 3×3 matrix. This is the sum of the three eigenvalues. Now each of these eigenvalues must be ± 1 since the elements of Γ_2 have order 2. So we can infer that the eigenvalues are $1, -1, -1$. The zero entries in the last row are each the sum of 3 cube roots of unity. The only way to get a zero sum from 3 cube roots of unity is to take exactly one of each. So we can infer that the 3 × 3 matrices that arise here must have distinct eigenvalues 1, ω and ω^2.

χ_1 is clearly the trivial character. The regular character: [12, 0, 0, 0] is expressible as a sum of irreducible characters as $\chi_1 + \chi_2 + \chi_3 + 3\chi_4$.

As can be seen a considerable amount of information about the group (and the representations themselves) can be recovered from the character table. Of course one has to know something about the group in the first place to be able to construct the character table. But we can learn new things about a group by using characters.

§ 7.7. Examples of Character Tables

Example 15: The character table of the trivial group 1 is:

class	**1**
size	**1**
χ_1	1
order	**1**

Example 16: The character table of $\mathbf{C}_2 = \langle A | A^2 = 1 \rangle$ is:

class	**1**	**A**
size	**1**	**1**
χ_1	1	1
χ_2	1	−1
order	**1**	**2**

Since the group generated by the integer –1 is $\mathbf{C}_2$, we have a representation by 1×1 matrices: $1 \to (1)$, $A \to (-1)$ and the character of this linear character is χ_2. Since the sum of squares of the degrees of χ_1 and χ_2 is 2, the order of $\mathbf{C}_2$, these are the only irreducible characters.

Example 17: The character table of $\mathbf{C}_3 = \langle A|A^3 = 1\rangle$ is:

class	1	A	A^2
size	1	1	1
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω
order	1	3	3

Clearly the character tables of cyclic groups are easily calculated. For $\mathbf{C}_n$ there are n linear characters, expressible in terms of n'th roots of unity.

Example 18:

The character table of $\mathbf{V_4} = \langle A, B|A^2 = B^2 = 1, BA = AB\rangle$ is:

class	1	A	B	AB
size	1	1	1	1
χ_1	1	1	1	1
χ_2	1	1	–1	–1
χ_3	1	–1	1	–1
χ_4	1	–1	–1	1
order	1	2	2	2

Viewed as 1×1 matrices these are clearly representations. In the next chapter we will see how to manufacture the character table of a direct product out of the character tables of the factors and, as a result, we will be able to easily calculate the character table of any finite abelian group. So let us turn our attention to the smallest non-abelian group.

Example 19: The character table of $\mathbf{S_3}$ is:

class	**1**	**(×××)**	**(××)**
size	**1**	**2**	**3**
χ_1	1	1	1
χ_2	1	1	–1
χ_3	2	–1	0
order	**1**	**3**	**2**

χ_1 is, as usual, the trivial character.
χ_2 is the character that maps even permutations to 1 and odd permutations to –1.

By the Fundamental Theorem of characters, since there are only 3 conjugacy classes, there must be only 3 irreducible characters. So χ_3 has to be found.

Suppose the degree of $\chi_3 = n$. Since the sum of squares of the degrees has to total the group order, we have $1^2 + 1^2 + n^2 = 6$ and so $n = 2$. Using orthogonality of columns we can complete the table as above.

Here we’ve used fairly simple techniques. For larger groups we need to develop more advanced techniques, which we do in the next chapter.

EXERCISES FOR CHAPTER 7

EXERCISE 1:
Examine the following character table for a finite group G and answer the following questions. Give adequate reasons for your answers.

	Γ_1	Γ_2	Γ_3	Γ_4
χ_1	1	1	1	1
χ_2	3	–1	0	0
χ_3	1	1	ω	ω^2
χ_4	1	1	ω^2	ω

(a) What is |G|?
(b) Find the sizes of the conjugacy classes.
(c) Find the orders of the kernels of each of the corresponding irreducible representations.
(d) Which of the irreducible characters are faithful?
(e) Find the order of the elements in each conjugacy class.

EXERCISE 2: Complete the following character table, giving brief explanations as to how each entry is obtained.

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
χ_1					
χ_2	1	i	1		–1
χ_3		0	–1	0	0
χ_4	1		1	–1	1
χ_5	1	–i	1		–1

EXERCISE 3: For the character table obtained in exercise 2, compute the size and the order of the elements of each of the conjugacy classes.

EXERCISE 4: Examine the following character table for a finite group G and answer the following questions. Give adequate reasons for your answers.

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1	-1
χ_3	1	1	ω	ω^2	1	ω	ω^2
χ_4	1	1	ω^2	ω	1	ω^2	ω
χ_5	1	1	ω	ω^2	-1	$-\omega$	$-\omega^2$
χ_6	1	1	ω^2	ω	-1	$-\omega^2$	$-\omega$
χ_7	6	-1	0	0	0	0	0

(a) What is $|G|$?
(b) Find the sizes of the conjugacy classes.
(c) Find the orders of the kernels of each of the irreducible representations.
(d) Which of the irreducible representations are faithful?
(e) Draw the lattice diagram for all the normal subgroups of G.
(f) Find Z(G) and G′. For each of them identify which conjugacy classes they are built up from and give a well-known group that it is are isomorphic to.
(g) Find the order of the elements in each conjugacy class.

(h) Express the following character as a sum of irreducible characters:

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7
χ	14	7	2	2	–6	0	0

EXERCISE 5: Examine the following character table for a finite group G and answer the following questions. Give adequate reasons for your answers.

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8	Γ_9
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	–1	–1	1	1
χ_3	–1	2	2	–1	–1	0	0	–1	2
χ_4	1	1	1	–1	1	–1	1	–1	–1
χ_5	1	1	1	–1	1	1	–1	–1	–1
χ_6	–1	2	2	1	–1	0	0	1	–2
χ_7	–2	–2	2	0	2	0	0	0	0
χ_8	1	–2	2	$\sqrt{3}i$	–1	0	0	$-\sqrt{3}i$	0
χ_9	1	–2	2	$-\sqrt{3}i$	–1	0	0	$\sqrt{3}i$	0

(a) How many conjugacy classes does G have?
(b) Which conjugacy class is {1}?
(c) What is |G|?
(d) Find the sizes of the conjugacy classes.
(e) Find the orders of the kernels of each of the irreducible representations.
(f) Which of the irreducible representations are faithful?

(g) Draw the lattice diagram for all the normal subgroups of G.

(h) Find Z(G) and G′. Identify which conjugacy classes they are built up from and describe a well-known group that they are isomorphic to.

(i) How many of the elements of G have order 3?

EXERCISE 6: Complete the following character table, giving brief explanations as to how each entry is obtained.

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
size		**3**	**3**	**16**	**3**	**3**		**3**
χ_1			-1			-1		-1
χ_2					1			
χ_3	1		1	ω	1	1		1
χ_4	1		1	ω^2	1	1		1
χ_5	3		$-1+2i$		-1	1		$-1-2i$
χ_6								
χ_7	3		1		-1	$-1-2i$		1
χ_8					-1	$-1+2i$		
order	**1**		**4**	**3**	**2**	**4**		**4**

EXERCISE 7: Complete the following character table, giving brief explanations as to how each entry is obtained.

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7
size	**1**	**1**	**6**	**4**	**4**	**4**	**4**
χ_1							
χ_2	1	1	1	ω		ω^2	
χ_3	1	1	1	ω^2			
χ_4	2			$-\omega$			
χ_5	2			$-\omega^2$			
χ_6						1	
χ_7							
order	**1**		**4**	**3**			

SOLUTIONS FOR CHAPTER 7

EXERCISE 1: (a) 12; (b) 1, 3, 4, 4; (c) $|\ker \rho_1| = 12$, $|\ker \rho_2| = 1$, $|\ker \rho_3| = |\ker \rho_4| = 4$; (d) χ_2; (e) 1, 2, 3, 3.

EXERCISE 2: None of χ_2 to χ_5 are the trivial character so χ_1 must be trivial. Since *i* is not real its conjugate $-i$ must appear in that row, so $\Gamma_4 = \Gamma_2^{-1}$. We can therefore complete columns 2 and 4. By orthogonality of columns 1 and 3 we deduce that $\deg \chi_3 = 4$. The character table is thus:

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
χ_1	1	1	1	1	1
χ_2	1	i	1	–i	–1
χ_3	4	0	–1	0	0
χ_4	1	–1	1	–1	1
χ_5	1	–i	1	i	–1

EXERCISE 3: The group has order 20. We can now compute the sizes of the conjugacy classes: 1, 5, 4, 5, 5. Since the order is even the group must contain elements of order 2. Their characters must be real so the elements of order 2 must lie in Γ_3 or Γ_5 or both. But the centraliser of an element in Γ_3 has order 5, so the elements of order 2 must lie in Γ_5. Also since the group order, 20, is divisible by 5 there must be elements of order 5. Clearly these can't lie in Γ_2 or Γ_4 since i has order 4. So they must lie in Γ_3.

Of course the only element of Γ_1 has order 1, so that just leaves Γ_2 and Γ_4. Since $\Gamma_4 = \Gamma_2^{-1}$ they must all have the same order. This must divide 20 and, since i has order 4, their order must be divisible by 4. Thus they have order exactly 4. The orders of the elements of the conjugacy classes are thus 1, 4, 5, 4, 2 respectively.

EXERCISE 4:

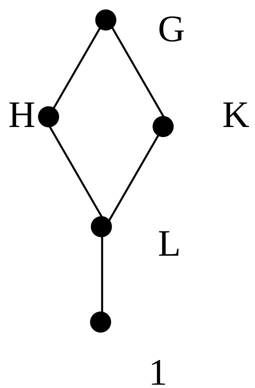
(a) 42;

(b)1, 6, 7, 7, 7, 7, 7

(c) 42, 21, 14, 7, 7, 1;

(d) χ_7

(e)



$H = \Gamma_1 + \Gamma_2 + \Gamma_5$, $K = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$, $L = \Gamma_1 + \Gamma_2$

(f) $Z(G) = 1$, $G' = L \cong \mathbf{C}_7$

(g) Since $L \cong C_7$ the elements of Γ_2 have order 7. Since $|G| = 42$ there must be elements of order 2, 3. The only class that could contain elements of order 2 is Γ_5. By considering the linear characters we see that the order of the elements of Γ_6, Γ_7 is a multiple of 6. The only

multiples of 6 dividing 42 are 6 and 42 and G is clearly not cyclic. So the elements of Γ_6, Γ_7 must have order 6 leaving the elements of Γ_3, Γ_4 being the ones of order 3. (Note that since $\Gamma_4 = \Gamma_3^{-1}$ and $\Gamma_7^{-1} = \Gamma_6$ the elements in each of these pairs of conjugacy class have the same order.)
The elements of have orders 1, 7, 3, 3, 2, 6, 6

(h) $\chi = m_1\chi_1 + ... + m_7\chi_7$ where $m_i = \langle\chi|\chi_i\rangle$ so

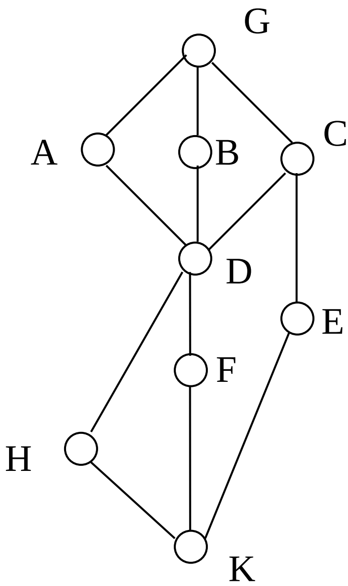
$$\chi = \chi_1 + 3\chi_2 + 2\chi_5 + 2\chi_6 + \chi_7.$$

EXERCISE 5:

(a) 9 conjugacy classes;
(b) Γ_3 (largest modulus);
(c) $|G| = \Sigma n_i^2 = 24$.
(d) $|\Gamma_1| = |\Gamma_4| = |\Gamma_5| = |\Gamma_8| = |\Gamma_9| = 24/12 = 2$; $|\Gamma_2| = |\Gamma_3| = 24/24 = 1$; $|\Gamma_6| = |\Gamma_7| = 24/4 = 6$.
(Check: the sizes total 24.)
(e) $|\ker \rho_1| = 24$;
$|\ker(\rho_2)| = |\Gamma_1| + |\Gamma_2| + |\Gamma_3| + |\Gamma_4| + |\Gamma_5| + |\Gamma_8| + |\Gamma_9| = 12$;
$|\ker(\rho_3)| = 4$; $|\ker(\rho_4)| = |\ker(\rho_5)| = 12$; $|\ker(\rho_6)| = 2$;
$|\ker(\rho_7)| = 3$; $|\ker(\rho_8)| = |\ker(\rho_9)| = 1$.

(f) Only ρ_8 and ρ_9 are faithful.

(g)



$A = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_5 + \Gamma_6$
$B = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_5 + \Gamma_7$
$C = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5 + \Gamma_8 + \Gamma_9$
$D = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_5$, $E = \Gamma_2 + \Gamma_3 + \Gamma_9$, $F = \Gamma_3 + \Gamma_5$
$G = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5 + \Gamma_6 + \Gamma_7 + \Gamma_8 + \Gamma_9$
$H = \Gamma_2 + \Gamma_3$
$K = \Gamma_3$

(h) Z(G) is the union of all the classes of size 1 and consists of classes 2, 3. $Z(G) \cong \mathbf{C}_2$.
G′ is the intersection of the kernels of the linear representations and consists of classes 1, 2, 3 and 5. It has order 6 but is clearly not isomorphic to $\mathbf{S}_3$ (it has a normal subgroup of order 2) so it is isomorphic to $\mathbf{C}_6$.

(i) For an element *g* of order 3 the only possible eigenvalues for (ρ(*g*)) are 1, ω and ω^2. Hence these are the only possible values for linear characters χ. By inspecting the table we see that the only possibilities for elements of order 3 are Γ_1, Γ_2, Γ_3 and Γ_5. But Γ_3 ={I} and the elements of Γ_2 are in a subgroup of order 2. Now Γ_5, being in a subgroup of order 3 must consist of 2 elements of order 3. These account for all the elements of order 3 in $G' \cong \mathbf{C}_3$ and so Γ_1 must consist of the 2 elements of order 6.

EXERCISE 6: The conjugates of χ_5 must be χ_6 and the conj of χ_7 must be χ_8. χ_2 must be the trivial character. $\Gamma_4^{-1} = \Gamma_7$; $|\Gamma_1| = 1$, $|G| = 48$, deg $\chi_1 = 3$, so the remaining entries in Γ_4, Γ_7 are 0, $\Gamma_6^{-1} = \Gamma_2$, $\chi_1(\Gamma_5) = 3$ by orthogonality with Γ_1.

	Γ_1	**Γ_2**	**Γ_3**	**Γ_4**	**Γ_5**	**Γ_6**	**Γ_7**	**Γ_8**
size	**1**	**3**	**3**	**16**	**3**	**3**	**16**	**3**
χ_1	3	–1	–1	0	3	–1	0	–1
χ_2	1	1	1	1	1	1	1	1
χ_3	1	1	1	ω	1	1	ω^2	1
χ_4	1	1	1	ω^2	1	1	ω	1
χ_5	3	1	–1+2i	0	–1	1	0	–1–2i
χ_6	3	1	–1–2i	0	–1	1	0	–1+2i
χ_7	3	–1+2i	1	0	–1	–1–2i	0	1
χ_8	3	–1–2i	1	0	–1	–1+2i	0	1
order	**1**	**4**	**4**	**3**	**2**	**4**	**3**	**4**